Computing a simple stochastic OLG example

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Outline

Now we consider the RCE in a specific example with closed-form solution.

Example

Assume that the capital depreciation rate $\delta = 0$. Let, as before,

$$A_t F(K_t, L_t) = A_t K_t^{\alpha} N_t^{1-\alpha}, \quad \alpha \in (0, 1),$$

and

$$u(c) = \ln(c),$$

and

$$\ln(A_{t+1}) = (1-\gamma)\ln(A) + \gamma \ln(A_t) + \epsilon_{t+1}, \quad A > 0, \gamma \in (0, 1).$$

Example (continued)

We showed, consumer's optimal intertemporal consumption *contingent plans* satisfy:

$$\frac{1}{c_t^t} = \beta \mathbb{E}_{\mu,t} \left\{ \left[\frac{1 + r_{t+1}}{c_{t+1}^t} \right] \right\}$$

Using consumer's budget constraints, and using capital market clearing condition, this can be re-written as

$$\frac{1}{w_t - (1+n)k_{t+1}} = \beta \mathbb{E}_{\mu,t} \left\{ \left[\frac{1 + r_{t+1}}{(1 + r_{t+1})(1+n)k_{t+1}} \right] \right\}$$

Example (continued)

Simplify RHS to get

$$\frac{1}{w_t - (1+n)k_{t+1}} = \beta \mathbb{E}_{\mu,t} \left\{ \left[\frac{1}{(1+n)k_{t+1}} \right] \right\}$$

so the only stochastic element $r_{t+1} := r_{t+1}(k_{t+1}, A_{t+1})$ dropped out from the first-order condition.

Since k_{t+1} is known at time t, the condition also holds "within" the expectations operator:

$$\frac{1}{w_t - (1+n)k_{t+1}} = \beta \left[\frac{1}{(1+n)k_{t+1}} \right]$$

Example (continued)

Re-arrange for k_{t+1}, we have

$$k_{t+1} = \frac{\beta}{(1+n)(1+\beta)} w_t.$$

Looks just like its deterministic cousin we derived earlier, hey?

Not quite! Now, from firm's optimal labor demand (**??**), we have

$$w_t = (1-\alpha) A_t k_t^{\alpha}$$

which suffers from random perturbations by A_t! So we have a stochastic difference equation solution to the RCE of this example:

$$k_{t+1} = \frac{\beta}{(1+n)(1+\beta)} (1-\alpha) A_t k_t^{\alpha}.$$

Example

The solution to this economy's RCE beginning from (k_0, A_0) is a contingent allocation plan (sequence of decision functions), $(k_{t+1}, c_t, c_{t+1}^t, y_t)(k_t, A_t)$ supported by state-contingent prices $(w_t, r_t)(k_t, A_t)$ that satisfies

1. $k_{t+1} = \frac{\beta}{(1+n)(1+\beta)}(1-\alpha)A_t k_t^\alpha$,
2. $w_t = (1-\alpha)A_t k_t^\alpha$,
3. $r_t = \alpha A_t k_t^{\alpha-1} - \delta$,
4. $c_t = w_t - (1+n)k_{t+1}$,
5. $c_{t+1}^t = (1+r_{t+1})k_{t+1}$,
6. $y_t = A_t k_t^\alpha$,
7. $i_t = s_t = (1+n)k_{t+1}$ (investment flow).

for every possible random history of TFP levels, $\{A_t\}_{t=0}^\infty$.

Exercise

- *Choose your parameter values* $(\alpha, \beta, \delta, A, \gamma, M)$.
- *Generate a random sequence* $\{A_t\}_{t=0}^{\infty}$ *according to the law of motion*

$$\ln(A_{t+1}) = (1-\gamma)\ln(A) + \gamma\ln(A_t) + \epsilon_{t+1} \equiv H(A_t, \epsilon_{t+1}).$$

For example, assume $\epsilon_{t+1} \sim U[-(1-\gamma)\ln(A), \ln(M)]$.

- *To generate a uniformly distributed random variable on an interval* $[a, b]$, *use the Excel command,* `RAND()*(b - a) + a`. *[More on Excel]*
- *Use Excel or any spreadsheet software to calculate a sample RCE path using these equations:*
  1. $k_{t+1} = \frac{\beta}{(1+n)(1+\beta)}(1-\alpha)A_t k_t^{\alpha} \equiv G(A_t, k_t)$,
  2. $w_t = (1-\alpha)A_t k_t^{\alpha}$,
  3. $r_t = \alpha A_t k_t^{\alpha-1} - \delta$,
  4. $c_t = w_t - (1+n)k_{t+1}$,
  5. $c_{t+1}^{t} = (1+r_{t+1})k_{t+1}$,
  6. $y_t = A_t k_t^{\alpha}$,
  7. $i_t = s_t = (1+n)k_{t+1}$ *(investment flow).*

Pseudocode

A stylized recipe for computing a sample stochastic RCE path. First generate a random sample stored as a vector, $\mathbf{e} = \{\epsilon_t\}_{t=0}^{T}$.

Example (Matlab)

If $\mathbf{e}$ is to be a uniformly distributed random vector, use the following command:

randyolg.m

```
a = -(1-gamma)*log(A);
b = log(M);
e = zeros(T+1,1);              % pre-allocate memory
e = a + (b-a).*rand(T+1,1);  % populate null vector with uniform r.v.'s
omega = exp(e);         % take anti-log of e
```

Figure: Generating a vector containing $T+1$ observations of uniform random variables.

Pseudocode

ALGORITHM 1. Simulating sample RCE outcomes

Input: (A_0, k_0), Equilibrium system: (H, G), random sample $\omega \leftarrow \{\exp(\epsilon_t)\}_{t=0}^{T}$, Null vectors $\mathbf{A}(:) = \mathbf{k}(:) = \mathbf{0}_{(T+1)\times 1}$.

```
set
    A(1) ← A_0
    k(1) ← k_0
    t ← 1
end
while t ≤ T do
    A(t+1) ← H[A(t), ω(t+1)]
    k(t+1) ← G[A(t), k(t)]
    set
        t ← t + 1
    end
```

$$
\begin{aligned}
&\textbf{set}\\
&\quad \mathbf{A}(1) \leftarrow A_0\\
&\quad \mathbf{k}(1) \leftarrow k_0\\
&\quad t \leftarrow 1\\
&\textbf{end}\\
&\textbf{while } t \leq T \textbf{ do}\\
&\quad \mathbf{A}(t+1) \leftarrow H[\mathbf{A}(t), \omega(t+1)]\\
&\quad \mathbf{k}(t+1) \leftarrow G[\mathbf{A}(t), \mathbf{k}(t)]\\
&\quad \textbf{set}\\
&\quad\quad t \leftarrow t+1\\
&\quad \textbf{end}
\end{aligned}
$$

Output: Random sample RCE path $\{k_{t+1}\}_{t=0}^{T} \leftarrow \mathbf{k}$

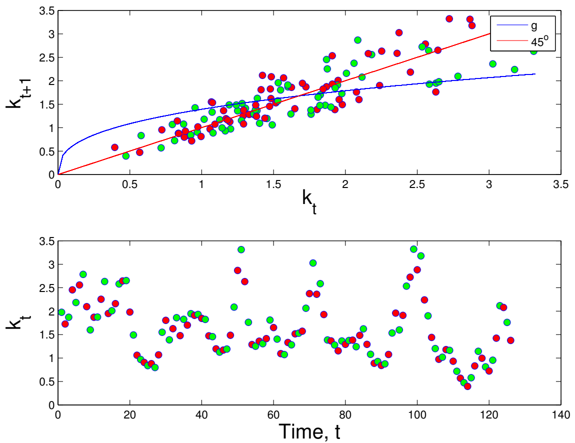

Figure: Sample RCE path for capital now appears as random fluctuations around the deterministic steady state where $k_{t+1} = g(A_t, k_t)$, because A_t is a stochastic process.

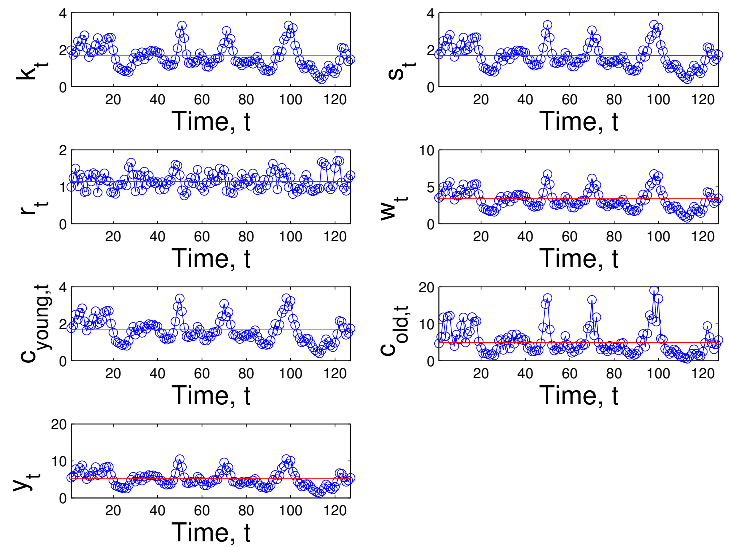

Figure: Sample RCE path for other variables which are functions of (A_t, k_t).

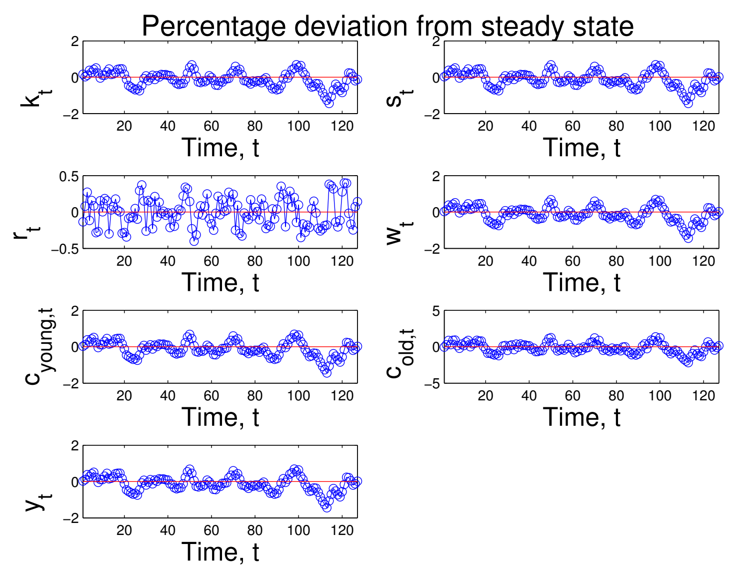

Figure: Sample RCE path for same variables transformed as percentage deviations from respective steady state values. E.g. $\ln(y_t/y_{ss})$.

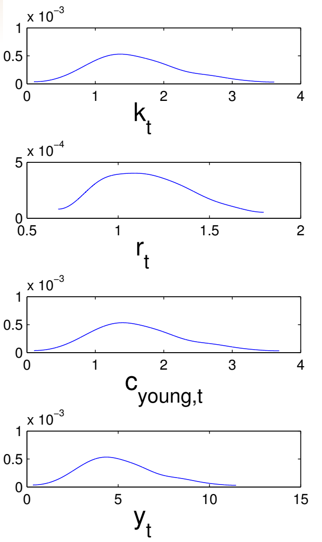

Figure: Sample distribution of RCE outcomes over time. By law of large numbers this approaches the stationary distribution (with probability 1).

Table: Mean for simulated data and steady state values

Variable	simulated	theory
s_t	1.632	1.693
w_t	3.280	3.402
$c_{young,t}$	1.648	1.710
y_t	5.125	5.316

Notice that the simulated RCE means are quite close to the theoretically calculated steady state values? If I increase the simulation sample observations, they should be the same, w.p.1.

Remark

1. *Example illustrates techniques in stochastic modeling and a simple economic (OLG) theory for generating theory consistent "business cycles".*
2. *Should we take this model seriously as a quantitative model of real business cycles?*
3. *Not quite. OLG model designed for long run analysis. Notion of a "period t" is very long. At business cycle frequencies, a period t usually is one quarter (3 months).*
4. *Mismatch between theory and measurement. Need a model with more refined notion of generations. Or more generally, assume infinitely lived agents.*
5. *Given calibration of parameters, model does not generate realistic business cycle facts ....*

- In principle, as we saw, the model implies an equilibrium statistical process.
- We saw the sample distribution of the model variables.
- We can use these simulated data to calculate the relevant business cycle statistics – e.g. standard deviations, means, correlations with output (measure of procyclicality), etc.
- But this model is no good, quantitatively.

Table: Standard deviations for simulated data

Variable	std
i_t	0.635
w_t	1.277
$c_{young,t}$	0.642
y_t	1.995