OLG: Steady-state Optimality and Competitive Equilibrium

Timothy Kam

Research School of Economics
Australian National University

ECON8026, This version April 17, 2018
Outline

1 Overview

2 Competitive equilibrium
   - Log-utility example

3 Long-run Optimality
   - Long-run feasibility
   - Golden rule
   - Long-run feasibility
Overview

This lecture:

- Restate the recursive competitive equilibrium (RCE) characterization
- Welfare properties of RCE in the OLG model (in a steady state)
- (Steady state) Competitive equilibrium of OLG model may/may not be Pareto optimal
**Definition**

Given \( k_0 \), a RCE is a price system \( \{w_t(k_t), r_t(k_t)\}_{t=0}^{\infty} \) and allocation \( \{k_{t+1}(k_t), c_t(k_t), c_{t+1}(k_t)\}_{t=0}^{\infty} \) that satisfies, for each \( t \in \mathbb{N} \):

1. **Consumer’s lifetime utility maximization:**

   \[
   \beta \frac{U_c(c_{t+1})}{U_c(c_t)} = \frac{1}{1 + r_{t+1}}, \quad \text{and,} \quad c_t + \frac{c_{t+1}}{(1 + r_{t+1})} = w_t \cdot 1.
   \]

2. **Firm’s profit maximization:**

   \[
   f'(k_t) = r_t + \delta, \quad \text{and,} \quad f(k_t) - k_t f'(k_t) = w_t.
   \]

3. **Market clearing in the credit/capital market:**

   \[
   (1 + n)k_{t+1} = (w_t \cdot 1 - c_t).
   \]
Recursive competitive equilibrium ...

Young-age budget constraint:

\[ c_t^t = w_t - s_t \]
\[ = [f(k_t) - k_t f'(k_t)] - s_t \equiv w(k_t) - s_t \]

and old-age budget constraint:

\[ c_{t+1}^t = (1 + r_{t+1}) s_t \]
\[ = [f_k(k_{t+1}) + 1 - \delta] s_t \equiv R(k_{t+1}) \cdot s_t \]
Recursive competitive equilibrium (cont’d) ...

Let $R_{t+1} := R(k_{t+1})$. From Euler equation, denote for all $t \in \mathbb{N}$:

$$E(s_t, w_t, R_{t+1}) \equiv -U_c(w_t - s_t) + \beta R_{t+1} U_c(R_{t+1}s_t) = 0,$$

In words, we have:

- a necessary sequence of FOC’s (Euler equations) characterizing the optimal savings trajectory $\{s_t\}_{t=0}^\infty$ (of all generations);
- Given (i.e. taken as parametric by consumer) market terms of trades $(w_t, R_{t+1})$, this Euler equation implicitly defines the solution as some function $s : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ such that $s_t = s(w_t, R_{t+1})$. 
Recall assumptions on primitive $U$:

- $U$ is continuous on $\mathbb{R}_+$
- For all $c > 0$, $U_c(c) > 0$, and, $U_{cc}(c) < 0$ exist
- $\lim_{c \searrow 0} U_c(c) = +\infty$

Then the function $(w, R) \mapsto s(w, R)$, such that

$$s_t = s(w_t, R_{t+1})$$

is well-defined and $s_w(w, R)$, and $s_R(w, R)$ exist for every $(w, R) \in \mathbb{R}_+^2$. 
Definition (IES)

Given per-period utility function $U$, the intertemporal elasticity of substitution, evaluated at a point $c$ is

$$\sigma(c) = - \frac{U_c(c)}{U_{cc}(c)} \cdot c$$

Remark: Note similarity to Arrow-Pratt measure of relative risk aversion? How?
Recursive competitive equilibrium (cont’d) ...

From Euler equation (dropping $t$ subscripts),

$$E(s, w, R) \equiv -U_c(w - s) + \beta RU_c(Rs) = 0,$$

We can use the implicit function theorem to obtain:

$$E_s ds + E_w dw + E_R dR = 0,$$

where:

- $E_s := \frac{\partial E(s, w, R)}{\partial s} = U_{cc}(w - s) + \beta R^2 U_{cc}(Rs) < 0$
- $E_w := \frac{\partial E(s, w, R)}{\partial w} = -U_{cc}(w - s) > 0$
- $E_R := \frac{\partial E(s, w, R)}{\partial R} = \beta U_c(Rs) \left[1 - \frac{1}{\sigma(Rs)} \right] \overset{\leq}{> 0}$
Hold $R$ constant (i.e. $dR = 0$), we have

$$s_w(w, R) = -\frac{E_w}{E_s} = \left[ 1 + \frac{\beta R^2 U_{cc}(Rs)}{U_{cc}(w - s)} \right]^{-1} \in (0, 1);$$

i.e. the marginal propensity to save out of $w$ (equiv. lifetime income) is

- endogenous, and depends (in general) on aggregate state (relative prices) $(w, R)$,
- is bounded in the set $(0, 1)$. Why? Because $(c^t_t, c^t_{t+1})$ are normal goods!
Hold $w$ constant (i.e. $dw = 0$), we have

$$s_R(w, R) = -\frac{E_R}{E_s}$$

$$= -\frac{\beta U_c(Rs)[1 - 1/\sigma(Rs)]}{U_{cc}(w - s) + \beta R^2 U_{cc}(Rs)} \geq 0, \text{ if } \sigma(Rs) \geq 0.$$

i.e. effect of the rate of return on capital on saving:

- is ambiguous ...
- depends on $\sigma(Rs) \geq 0$, and therefore on specification of $U$.  

Given an optimal savings rule (equiv. consumption demand functions), \( s(w(k_t), R(k_{t+1})) \), a RCE sequence of allocations \( \{s_t, c^t_t, c^t_{t+1}, k_{t+1}\}_{t \in \mathbb{N}} \) satisfies for all \( t \in \mathbb{N} \):

- \( s_t = s(w(k_t), R(k_{t+1})) \),
- \( (1 + n)k_{t+1} = s_t \),
- \( c^t_t = w(k_t) - s_t \), and
- \( c^t_{t+1} = R(k_{t+1})s_t \),

for \( k_0 > 0 \) given.
Specific Example

Exercise

1. Derive, and therefore, show that $s(w, R)$ does not depend on $R$ in the case of $U(c) = \ln(c)$.
2. Explain why this is the case. Hint: You have learned this in consumer theory from intermediate microeconomics.
3. Depict this in the $(c_t, c_{t+1})$-space using the geometric devices of indifference and budget sets.
Optimality: steady states

**Focus:** long-run steady state.

We’ll study this in three successive components:

- Long-run feasibility
- Long-run maximal consumption: the Golden Rule
- Optimal long-run: Diamond’s “Golden Age”
Long-run feasibility I

Consider a long run (steady state), where per worker capital is $k$.

**Definition (Long-run feasibility)**

A steady-state $k \geq 0$ is feasible if net production at $k$ is non-negative:

$$\phi(k) := f(k) - (\delta + n)k \geq 0.$$ 

**Notes:**

- $f(k)$: gross output at a steady state $k$
- $(\delta + n)k$: claims on gross output at $k$
Recall assumption:

- $f$ continuous on $\mathbb{R}_+$
- $f_k(k) > 0$, $f_{kk}(k) < 0$ for all $k \in \mathbb{R}_+$
- $f$ satisfies Inada conditions ... (What are they?!)
Long-run feasibility III

Since $f_k(k) > 0$, $f_{kk}(k) < 0$ for all $k \geq 0$, then:

- $\phi_k(k) = f_k(k) - (\delta + n) \leq 0$,
- $\phi_{kk}(k) = f_{kk}(k) < 0$;

so that $\phi(k)$ is strictly concave.

Also note that:

- $\phi(0) = f(0) \geq 0$,
- $\lim_{k \searrow 0} \phi_k(k) = \lim_{k \searrow 0} f_k(k) - (\delta + n)$, and
- $\lim_{k \nearrow \infty} \phi_k(k) = \lim_{k \nearrow \infty} f_k(k) - (\delta + n)$. 
**Long-run feasibility IV**

**Long-run feasible sets:** If ...

**F1.** \( \phi_k(k) > 0 \), for all \( k \geq 0 \), any \( k \in \mathbb{R}_+ \) is long-run feasible.

**F2.** \( \phi_k(k) < 0 \), for all \( k \geq 0 \), and,

(a) if \( f(0) > 0 \), then \([0, \hat{k}]\) is long-run feasible, for some \( \hat{k} \in (0, \infty) \).

(b) if \( f(0) = 0 \), then only \( k = 0 \) is long-run feasible.

**F3.** \( \phi(k) \) non-monotonic. ...

... And \( \exists \tilde{k} \in (0, \infty) \) s.t. \( f(\tilde{k}) - (\delta + n)\tilde{k} = 0 \), then any \( k \in (0, \tilde{k}) \), is long-run feasible.
Exercise (Long-run-feasible sets of $k$)

Given assumptions about $f_k > 0$, $f_{kk} < 0$, and $f(0) \geq 0$, illustrate (in two respective diagrams) the graphs of:

1. $k \mapsto f(k)$ and $k \mapsto (\delta + n)k$, and therefore,
2. $k \mapsto \phi(k)$;

and show the corresponding long-run feasible sets, if F1, F2, or F3 were to hold.
Long-run feasibility VI

Exercise (Long-run-feasible sets of $k$ (cont’d))
The golden rule I

Consider cases:

**F1.** If 
\[
\lim_{k \to \infty} f_k(k) \geq (\delta + n) \Rightarrow \lim_{k \to \infty} \phi_k(k) > 0,
\]
then \(\phi\) is strictly increasing on \(\mathbb{R}_+\).

**F2.** If \(\phi_k(k) > 0\), then \(\phi\) is strictly decreasing. Not interesting — largest net production is at \(k = 0\): \(\phi(0) \geq 0\).

**F3.** If 
\[
\lim_{k \to \infty} f_k(k) < (\delta + n) < f_k(0)
\]
then \(\phi\) is non-monotonic:
- There exists a unique \(k_{GR} \in (0, \infty)\) such that \(\phi_k(k_{GR}) = 0\): i.e. net production is maximized, and
- \(\phi\) is increasing on \((0, k_{GR})\) and decreasing on \((k_{GR}, \infty)\).
The golden rule II

Proposition (Golden rule)

Assume production function $f$ such that
$\lim_{k \to \infty} f_k(k) < (\delta + n) < f_k(0)$.

Then there exists a unique $k_{GR} \in (0, \infty)$ such that
$\phi_k(k_{GR}) = f_k(k_{GR}) - (\delta + n) = 0$: i.e. net production is maximized.
The golden rule III

Exercise

*Illustrate the last proposition using appropriate diagrams.*
The golden rule IV

Exercise

Show that the regularity condition
\[ \lim_{k \to \infty} f_k(k) < (\delta + n) < f_k(0) \]
does not apply to the Cobb-Douglas family of functions \( f(\cdot; \alpha), \alpha \in (0, 1) \).

Remark: However, in Cobb-Douglas \( f(k; \alpha) = k^\alpha \) case with \( \alpha \in (0, 1), k_{GR} \in (0, \infty) \) still exists.
The golden rule V

**Remarks:** In any steady state $k$, ...
given regularity conditions on $U$ and $f$,

... we know from the RCE conditions, $(s_t, c^y_t, c^o_t)$ must converge to a well-defined limit $(s, c^y, c^o)$:

- savings function, $s = s(w(k), R(k))$,
- consumption (young), $c^y = w(k) - s$, and
- consumption (old), $c^o = R(k)s$. 
The golden rule VI

Therefore, the golden-rule proposition implies that there is a steady state golden-rule consumption level for each young and old agent, $(c^y_{GR}, c^o_{GR})$.

... The Solow-Swan golden-rule, per-se, says nothing about Pareto optimality in the long run! Why?

... What of steady state optimality in this model? Relation to the golden rule in this model?
Golden Age: optimal steady state I

Optimal steady state: “The Golden Age” (Diamond, 1965)

- Suppose we have the condition:
  \[ \lim_{k \to \infty} f_k(k) < (\delta + n) < f_k(0). \]
  This is guaranteed by the Inada conditions on \( f \).

- On a steady state path, \( k_t = k, c_t = c^y \) and \( c_{t+1} = c^o \) for all \( t \).

- The resource constraint is then:
  \[ f(k) = (\delta + n)k + c^y + (1 + n)^{-1}c^o. \]
Golden Age: optimal steady state II

- A Pareto allocation of consumption across periods of life along the steady state trajectory solves:

\[
\max_{(k,c^y,c^o) \in \mathbb{R}_+^3} \left\{ U(c^y) + \beta U(c^o) : f(k) = (\delta+n)k + c^y + (1+n)^{-1}c^o \right\}
\]

- This is still an intertemporal allocation problem, albeit stationary.
Golden Age: optimal steady state III

Characterization of Pareto-optimal steady state

1. The maximum feasible net production is attained when:

   \[ \phi_k(k) := f_k(k) - (\delta + n) = 0 \Rightarrow k = k_{GR}. \]

   (i.e. this is just the same condition characterizing the golden-rule per-worker capital stock, at steady state!)

2. Given assumption on \( f \) such that case F3 prevails, we then know \( k_{GR} \in (0, \infty) \).
Golden Age: optimal steady state IV

Also, the maximum of $U(c) + \beta U(c^o)$ s.t.

$$\phi(k) = c^y + (1 + n)^{-1} c^o$$

is characterized by:

$$\phi(k_{GR}) = c^y_{GR} + \frac{c^o_{GR}}{1 + n},$$

and,

$$U_c(c^y_{GR}) = \beta (1 + n) U_c(c^o_{GR}).$$
Golden Age: optimal steady state V

Proposition (Optimal steady state)

Given assumptions above, a unique Pareto-optimal steady state exists: \( k_{GR} \) satisfying

\[ f_k(k_{GR}) - (\delta + n) = 0; \]

(Golden rule)

and \( c^y_{GR} \) and \( c^o_{GR} \), respectively, satisfy

\[ \phi(k_{GR}) = c^y_{GR} + \frac{c^o_{GR}}{1 + n}, \]

(Resource constraint)

and,

\[ U_c(c^y_{GR}) = \beta(1 + n)U_c(c^o_{GR}). \]

(Euler equation)
Optimal vs. CE arbitrage I

If we decentralized previous Pareto planning problem ...

Given relative price (btw. young-vs-old consumption) $R_{t+1}$, each consumer’s optimal decisions $(c^t_t, c^t_{t+1})$ satisfy

$$U_c(c^t_t) = \beta R_{t+1} U_c(c^t_{t+1}).$$

(Euler eqn: at CE)
Optimal vs. CE arbitrage II

Optimal arbitrage: If \( R_{t+1} = (1 + n) \) for all \( t \), i.e. Samuelson’s (1958) “biological return” equals market terms of trade btw \((c^t, c^{t+1})\), so there is a triple \((c^y, c^o, k)\) such that

\[
U_c(c^y) = \beta(1 + n)U_c(c^o),
\]

and the actual value of lifetime expenditure on consumption (for each agent) is

\[
c^y + \frac{c^o}{1 + n} = w(k) = f(k) - f_k(k)k
\]

\[
= f(k) - (\delta + n)k.
\]
Optimal vs. CE arbitrage III

But ... at \( R = 1 + n \), market clearing at steady state requires

\[(1 + n)k = s[w(k), 1 + n] = w(k) - c^y.\]

If we impose the optimal allocation, setting \( k = k_{GR} \), then \( c^y = c^y_{GR} \) and \( c^o = c^o_{GR} \), in general,

\[(1 + n)k_{GR} \neq s[w(k_{GR}), 1 + n].\]

Optimal steady-state path, in general, not equivalent to the competitive equilibrium steady-state path.
Optimal vs. CE arbitrage IV

Proposition (Optimal allocation and life-cycle no-arbitrage)

The optimal steady state path \((k_{GR}, c^y_{GR}, c^o_{GR})\) satisfies:

- the decentralized no-arbitrage condition of each consumer where the return on saving is \(R = f_k(k_{GR}) + (1 - \delta) = 1 + n\); and

- her life-cycle income is \(w(k_{GR}) = f(k_{GR}) - f_k(k_{GR})k_{GR}\).

But her choice of saving is generally not equal to the level of Pareto-optimal invest: \(s[w(k_{GR}), 1 + n] \neq (1 + n)k_{GR}\).
Optimal vs. CE arbitrage V

To prove this, all we need is a counter-example.

Example ($\delta = 1$)

Let $U(c) = \ln(c)$ and $f(k) = k^\alpha$. Then

- $k_{GR} = \left[\alpha/(1 + n)\right]^{1/(1 - \alpha)}$
- $\phi(k_{GR}) = w(k_{GR}) = (1 - \alpha)k_{GR}^\alpha$
- $c^y_{GR} = (1 + \beta)^{-1}\phi(k_{GR})$
- $c^o_{GR} = (1 + \beta)^{-1}[(1 + n)\beta]\phi(k_{GR})$
- $s[w(k_{GR}), 1 + n] = \beta(1 + \beta)^{-1}\phi(k_{GR})$.

Show that at a steady state $k = k_{GR}$ it is possible that it is not consistent with a RCE.
Optimal vs. CE arbitrage VI

Example (cont’d)
Observe that:

\[ s[w(k_{GR}), 1 + n] \equiv (1 + n)k_{GR}, \]

if and only if:

\[ \frac{\beta}{(1 + \beta)}(1 - \alpha)k_{GR}^\alpha \equiv \alpha k_{GR}^\alpha \iff \frac{\beta}{1 + \beta} \equiv \frac{\alpha}{1 - \alpha}. \]

Given \( \alpha \), if \( \beta \) too large (agent’s too patient), then savings exceeds golden rule capital stock. Only in special case where \( \beta/(1 + \beta) = \alpha/(1 - \alpha) \), do the two equal.
What is the reasoning behind RCE allocation not necessarily being an optimal one?

- FWT states that a competitive equilibrium is also Pareto optimal, as long as there exist complete markets, agents are price-takers and preferences are locally non-satiated.
- This steady state analysis showed a breakdown of what is known as the First Welfare Theorem (FWT).
Optimal vs. CE arbitrage VIII

- The problem here is that in a CE each generation’s old agents do not care about the next generation’s young.
- The former eats up the total dividend from and the remainder of their capital stock.
- Competitive agents do not internalize the need of moving resources intertemporally across infinitely far generations.
- They only move private resources across time (through savings) insofar as it maximizes their own lifetime utilities.
Optimal vs. CE arbitrage IX

- A planner in an optimal steady state cares about every generation and maximizes the net production subject to that being feasible; and
- Planner allocates consumption intertemporally for each generation according to the biological rate of exchange.
- Pareto planner internalizes the effect of shifting resources across infinite sequences of generations; and
- planner’s optimal allocation is feasible w.r.t. resource constraint that holds over all $t \in \mathbb{N}$. 
Over/under accumulation of capital I

At a steady state $\bar{k}$ of an RCE:

- If $f_k(\bar{k}) > \delta + n$, then $\bar{k} < k_{GR}$ (under-accumulation).
- If $f_k(\bar{k}) < \delta + n$, then $\bar{k} > k_{GR}$ (over-accumulation).
Over/under accumulation of capital II

Note in both cases, for a given $\bar{k}$,

- the maximum life-cycle utility satisfies:
  \[ U_c(c^y) + \beta(1 + n)U_c(c^o) \]
  given net production fixed at $\phi(\bar{k})$,

... but ...

- the life-cycle utility at the competitive steady state satisfies:
  \[ U_c(\bar{c}^y) = \beta[f_k(\bar{k}) + 1 - \delta]U_c(\bar{c}^o). \]
Over/under accumulation of capital III

Implications:

- Competitive equilibrium over- or under-accumulation of $\bar{k}$ on a steady state path is not Pareto optimal.

- E.g. if $\bar{k} > k_{GR}$ (over-accumulation):
  - possible to increase total consumption by reducing $k$ to yield total resources per period $\phi(k)$ forever.
  - If $k$ reduces discretarily to $k_{GR}$ at some period, total consumption will be $\phi(k) + (k - k_{GR})(1 + n) > \phi(k)$. Total consumption in that period rises.
  - For continuation periods, the surplus is now $\phi(k_{GR})$ forever. But by definition of golden rule, $\phi(k_{GR}) > \phi(k)$. So total consumption forever is maximized.
  - Therefore total consumption for every generation can be increased at all dates by moving $k$ towards $k_{GR}$.