OLG and capital

Timothy Kam

Research School of Economics
Australian National University

ECON8026, This version April 17, 2018

Outline

Overview

This set of notes:

- A simple model that results in a Solow-Swan-like equilibrium map $k \mapsto g(k)$.
- Agents are maximizing agents. They interact through competitive markets.
- Notion of (dynamic) equilibrium now must encode optimal decisions, which are consistent with equilibrium market terms of trade.
- Derive the recursive competitive equilibrium (RCE) characterization

Assumptions

- Agents live for two periods. One period $\approx$ 30 years.
- Time is denumerable: $t \in \mathbb{N} : \{0, 1, 2, ...\}$.
- Population of current young N_t. Grows at deterministic rate n.
- $N_{t+1} = (1 + n)N_t$, with N_0 given.
- Production technology: $Y_t = F(K_t, N_t)$;
  - is constant returns to scale (homogeneous of degree 1).
  - Assume $F_K, F_N > 0$, $F_{KK}, F_{NN} > 0$ and $F_{K,N} > 0$.
  - $\lim_{N \searrow 0} F_N(\cdot, N) = +\infty$; and, $\lim_{K \searrow 0} F_K(K, \cdot) = +\infty$;
  - $\lim_{N \nearrow +\infty} F_N(\cdot, N) = 0$; and, $\lim_{K \nearrow +\infty} F_K(K, \cdot) = 0$;

Young Agent

Notation:

- Consumption at time t (subscript) of young agent born at time t (superscript): c_t^t.
- Saves s_t when young.
- Does not value leisure time. Inelastically supplies 1 (normalized) unit of labor to obtain labor income at wage rate w_t.
- Deterministically becomes old in $t+1$.
- Consumes c_{t+1}^t when old in $t+1$.

Young Agents: Preferences

- Young agent's per-period utility function is $c \mapsto U(c)$. Assume for $c \in \mathbb{R}_+$:
  - U is twice continuously differentiable, with $U_c(c) > 0$, and, $U_{cc} < 0$;
  - $\lim_{c \searrow 0} U_c(c) = +\infty$; and
  - $\lim_{c \nearrow +\infty} U_c(c) = 0$.
- Note: with example $U(c) = \ln(c)$ used in class, we will require that either $c \in \mathbb{R}_{++}$ (i.e. c is strictly positive), or there exists an exogenous lower bound: $U(0) = \underline{U}$.
- Preference representation – lifetime or total discounted utility:

$$U(c_t^t) + \beta U(c_{t+1}^t), \qquad \beta \in (0,1).$$

Time-t, young agent orders sequences $\{c_t^t, c_{t+1}^t\}$ according to this decision criterion function.

Young Agents: Feasibility

- Each young agent faces a known sequence of budget constraints. Decisions in terms of sequences $\{c_t^t, c_{t+1}^t\}$ must satisfy these constraints.
- When young:

$$c_t^t + s_t \leq w_t \cdot 1;$$

- When old:

$$c_{t+1}^t \leq (1 + r_{t+1})s_t,$$

where r_{t+1} is the rental rate on capital, determined in a competitive equilibrium.

Young Agents: Feasibility

Definition

For each young agent at t, given market relative prices (w_t, r_{t+1}), a feasible choice $\{c_t^t, c_{t+1}^t\}$ satisfies

$$c_t^t + s_t \leq w_t \cdot 1;$$

and

$$c_{t+1}^t \leq (1 + r_{t+1}) s_t;$$

or, alternatively, satisfies the consolidated budget constraint

$$c_t^t + \frac{c_{t+1}^t}{(1 + r_{t+1})} \leq w_t \cdot 1.$$

Young Agents: Optimal decisions

Each young agent at t, given market relative prices (w_t, r_{t+1}), picks $\{c_t^t, c_{t+1}^t\}$ to maximize

$$U(c_t^t) + \beta U(c_{t+1}^t), \qquad \beta \in (0,1).$$

subject to

$$c_t^t + \frac{c_{t+1}^t}{(1+r_{t+1})} \leq w_t \cdot 1.$$

Young Agents: Optimal decisions

The young agent's optimal lifetime decisions $\{c_t^t, c_{t+1}^t\}$ satisfy the necessary (and sufficient, given assumptions on U and convexity of budget set) conditions:

$$\beta \frac{U_c(c_{t+1}^t)}{U_c(c_t^t)} = \frac{1}{1 + r_{t+1}},$$

and

$$c_t^t + \frac{c_{t+1}^t}{(1 + r_{t+1})} = w_t \cdot 1.$$

Remarks:

- These conditions characterize the optimal demand functions $c_t^t = c_t^t(w_t, r_{t+1})$ and $c_{t+1}^t = c_{t+1}^t(w_t, r_{t+1})$;
- We can also deduce the optimal savings supply function $s_t = s(w_t, r_{t+1})$;

given market relative prices (w_t, r_{t+1}).

- The optimizing agent equate preference driven marginal rate of substitution in consumption across the two period to the market determined rate of exchange in consumption between the two periods.
- The optimizing agent also ensures that there is no waste: optimal decision is at the boundary of the lifetime budget set so that consolidated budget constraint is binding.

Exercise

Derive the agents' consumption demand and savings functions in the case where $U(c) = \ln(c)$.

Why does the saving function in this case not depend on r_{t+1} explicitly?

Who owns what? I

Assumptions:

- Young agents own labor, N_t (in total)
- Old agents own capital stock, K_t

Firm: Decision problem

Competitive firms choose inputs K_t and N_t to maximize per-period profits

$$\pi(K_t, N_t) = F(K_t, N_t) - w_t N_t - (r_t + \delta)K_t.$$

Price of a unit of homogenous good is normalized to 1.

Profit is total revenue less total labor wage bill, less capital rental and depreciation cost. Rental cost is real market value of renting capital at relative price r_t.

Firm: optimal decisions

Denote $k_t := K_t/N_t$. The FONC for profit maximization (in per-worker variable terms) is

$$\begin{aligned} f'(k_t) &= r_t + \delta \\ f(k_t) - k_t f'(k_t) &= w_t \end{aligned}$$

Note: We have done this before!

Exercise

Derive the profit maximizing conditions for the case where $F(K,N) = K^{\alpha}(AN)^{1-\alpha}$, $\alpha \in (0,1)$ *and* $A > 0$.

Market clearing: Accounting I

- Resource constraint must hold in equilibrium (no waste): total resources equals total use

$$F(K_t, N_t) + (1-\delta)K_t = K_{t+1} + C_t,$$

where $C_t = N_{t-1}c_t^{t-1} + N_t c_t^t$ is *total* consumption demand by old and young agents.

- If we assume old own capital stock then total old consumption is financed by capital income (from rental and leftover):

$$N_{t-1}c_t^{t-1} = (r_t + \delta)K_t + (1-\delta)K_t.$$

Market clearing: Accounting II

- The total young agents consumption is financed by labor income net of savings

$$N_t c_t^t = N_t(w_t \cdot 1 - s_t).$$

Market clearing: Accounting

- Using these three accounting conditions

$$C_t = N_{t-1}c_t^{t-1} + N_t c_t^t$$
$$= (r_t + \delta)K_t + (1-\delta)K_t + N_t(w_t \cdot 1 - s_t).$$

- Since $(r_t + \delta)K_t + w_t N_t = F(K_t, N_t)$ by F being homogeneous of degree 1, we have

$$C_t = F(K_t, N_t) + (1-\delta)K_t - N_t s_t.$$

- Finally, using this in the resource constraint, we get the market clearing condition

$$K_{t+1} = N_t s_t \Rightarrow k_{t+1} = \frac{s_t}{1+n}.$$

RCE

- Note that from firm's optimal demand for capital and labor, we can deduce that market prices (w_t, r_t) will be a function of the aggregate state variable k_t.
- Private agent optimal consumption demand and savings supply functions are functions of (w_t, r_t) and hence of k_t.
- Young agents supply of labor function does not depend on k_t since we assumed that utility does not admit labor/leisure time as an argument. This is not the case in general.
- We can next collect optimal decision rules and market clearing conditions together to consistent describe allocations and prices that are (recursive) competitive equilibrium outcomes.

Definition

Given k_0, a RCE is a price system $\{w_t(k_t), r_t(k_t)\}_{t=0}^{\infty}$ and allocation $\{k_{t+1}(k_t), c_t^t(k_t), c_{t+1}^t(k_t)\}_{t=0}^{\infty}$ that satisfies, for each $t \in \mathbb{N}$:

1. Consumer's lifetime utility maximization:

$$\beta \frac{U_c(c_{t+1}^t)}{U_c(c_t^t)} = \frac{1}{1 + r_{t+1}}, \quad \text{and,} \quad c_t^t + \frac{c_{t+1}^t}{(1 + r_{t+1})} = w_t \cdot 1.$$

2. Firm's profit maximization:

$$f'(k_t) = r_t + \delta, \text{ and, } f(k_t) - k_t f'(k_t) = w_t.$$

3. Market clearing in the credit/capital market:

$$k_{t+1} = \frac{(w_t \cdot 1 - c_t^t)}{1 + n}.$$

Exercise (On the W/board)

1. *Characterize explicitly a recursive competitive equilibrium for the economy where* $U(c) = \ln(c)$ *for* $c > 0$, *and* $F(K_t, N_t) = K_t^{\alpha}(AN_t)^{1-\alpha}$.
2. *Find the steady state per-worker capital stock,* k.
3. *Identify the parameter that indexes each agent's "patience". What does this actually mean in the context of the model?*
4. *What happens to the steady state* k *if agents are less patient?*
5. *What happens to the steady state* k *if population growth rate is higher?*