# LINEAR STOCHASTIC DIFFERENCE EQUATIONS

## SETUP

Consider a stochastic process for our state variables described by the canonical first-order linear stochastic difference equation (LSDE).[1] In some theories and applications (e.g. approximate linearized real or monetary business cycle models) that we will encounter, the solution to the (recursive competitive) equilibrium is approximately given by:

$$x_{t+1} = A_0 x_t + C w_{t+1}, \qquad w_{t+1} \sim \text{i.i.d.}(0, I) \tag{1}$$

The models may also induce a relationship between statistically observable variables $y_t$ and the state variables in the theory $x_t$ such that

$$y_t = G x_t. \tag{2}$$

These items in (1) and (2), including the distribution of the shocks $w_t$, define a Markov kernel for the economy. (How?)

Consider a small monetary policy model as example. The state vector and the observed vector may look like this

$$x_t = \begin{pmatrix} g_t \\ a_t \\ \pi_t \\ \tilde{y}_t \end{pmatrix}, y_t = \begin{pmatrix} \pi_t \\ \tilde{y}_t \\ r_t \end{pmatrix}.$$

where the three variables in $y_t$ having data counterparts, respectively, are inflation, output gap, and the nominal interest rate. The variables $(g_t, a_t)$ could be exogenous random variable that are Markov processes themselves and have the interpretation of exogenous forcing variables or serially-correlated shocks – e.g. respectively, a government spending (demand) shock and a technology (supply) shock.

## 1. MOMENTS OF THE LSDE

### 1.1 Covariance stationary mean

To ensure that $\{x_t\}$ is **covariance stationary** we require that $A_0$ be a stable matrix. Now take expectations on both sides of (3):

$$E x_{t+1} = A_0 E x_t.$$

Let $\mu_t := E x_t$. Then we can also write the above as:

$$\mu_{t+1} = A_0 \mu_t.$$

A **stationary mean** is one that satisfies $E x_{t+1} = E x_t = \mu$ which can be found as

$$(I - A_0)\,\mu = 0.$$

So in this case, since $A_0$ is a stable matrix, $\mu$ is a vector of zeros. (Why?) Note that we have

$$\mu_t = (A_0)^t \mu_0.$$

If $A_0$ is stable, then,

$$\lim_{t \to 0} \mu_t \to \mu.$$

[1]This note references Sections 2.1-2.3 in Ljungqvist and Sargent (2004).

Note that if the state transition law has a constant

$$x_{t+1} = \kappa + A_0 x_t + C w_{t+1}, \qquad w_{t+1} \sim \text{i.i.d.}(0, I) \tag{3}$$

then the **stationary mean** is given by

$$(I - A_0)\,\mu = \kappa \Rightarrow \mu = (I - A_0)^{-1}\kappa.$$

We can easily transform the variable so that the LSDE has no constant term $\kappa$. In that case, the expanded matrix $\tilde{A}_0$ will have some unit eigenvalue(s). (Why?)

### 1.2 Covariance stationary covariances

Given $\mu$ we compute stationary covariance matrix as:

$$E\,(x_{t+1} - \mu)\,(x_{t+1} - \mu)' = A_0 E\,(x_t - \mu)\,(x_t - \mu)'\,A_0' + CC'$$

since $E w_{t+1} w_{t+1}' = I$.

Define $\Sigma_x(0) = E\,(x_{t+1} - \mu)\,(x_{t+1} - \mu)' = E\,(x_t - \mu)\,(x_t - \mu)'$ as the stationary covariance matrix.

And $\Sigma_x(0)$ satisfies:

$$\Sigma_x(0) = A_0 \Sigma_x(0)\,A_0' + CC'$$

**Stationary covariance matrix** satisfies

$$\Sigma_x(0) = A_0 \Sigma_x(0)\,A_0' + CC'$$

This is a *discrete Lyapunov* equation in $\Sigma_x(0)$. We can in practice solve this as a recursion starting with an initial positive definite guess $\Sigma_x^0 = I$:

$$\Sigma_x^{s+1} = A_0 \Sigma_x^s A_0' + CC'$$

And since $A_0$ is stable $\lim_{s\to\infty} \Sigma_x^s \to \Sigma_x(0)$.

We can also compute **autocovariances** at different time intervals, $j$. We can write

$$\begin{aligned}
x_{t+j} &= A_0 x_{t+j-1} + C w_{t+j} \\
x_{t+j} - \mu_{t+j} &= A_0\,(x_{t+j-1} - \mu_{t+j-1}) + C w_{t+j} \\
&= A_0\,[A_0\,(x_{t+j-2} - \mu_{t+j-2}) + C w_{t+j-1}] + C w_{t+j} \\
&\;\;\vdots \\
&= A_0^j\,(x_t - \mu_t) + C w_{t+j} + A_0 C w_{t+j-1} + \ldots + A_0^{j-1} C w_{t+1}
\end{aligned}$$

Postmultiply with $(x_t - \mu_t)'$ and take expectations:

$$\begin{aligned}
E\,(x_{t+j} - \mu_{t+j})\,(x_t - \mu_t)' &= A_0^j E\,(x_t - \mu_t)\,(x_t - \mu_t)' \\
\Sigma_x(j) &= A_0^j \Sigma_x(0)\,.
\end{aligned}$$

We worked out $\Sigma_x(0)$. So autocovariance at interval $j$, $\Sigma_x(j)$ is a function of $A_0$ and $\Sigma_x(0)$. $\Sigma_x(j)$ is independent of $t$. It depends only on $j$.

Observation equation: $y_t = G x_t$. Then we can also find

$$\Sigma_y(j) = E\,(y_{t+j} - \mu_{y,t+j})\,(y_t - \mu_{y,t})' = G \Sigma_x(j)\,G'$$

Note: $C$ representing noise statistics does not appear in the calculation of $\Sigma_x(j)$ for all $j$.

## 2. Impulse response functions

Lag operator notation: $L x_{t+1} := x_t$.

Suppose eigenvalues of $A_0$ are less than 1. Write

$$x_{t+1} = A_0 x_t + C w_{t+1}$$

as

$$(I - A_0 L)\, x_{t+1} = C w_{t+1}$$

*Neumann expansion*

$$(I - A_0 L)^{-1} = I + A_0 L + A_0^2 L^2 + ...$$

Multiply both sides of

$$(I - A_0 L)\, x_{t+1} = C w_{t+1}$$

with $(I - A_0 L)^{-1}$ to get

$$x_{t+1} = \left(I + A_0 L + A_0^2 L^2 + ...\right) C w_{t+1} = \sum_{j=0}^{\infty} A_0^j C w_{t+1-j}$$

i.e. a finite lag VAR process has an infinite VMA representation.

Alternative moving average representation. Iterate

$$x_{t+1} = A_0 x_t + C w_{t+1}$$

forward from $t = 0$:

$$\begin{aligned} x_1 &= A_0 x_0 + C w_1 \\ x_2 &= A_0 x_1 + C w_2 = A_0^2 x_0 + C w_2 + A_0 C w_1 \\ &\vdots \\ x_t &= A_0^t x_0 + \sum_{j=0}^{t-1} A_0^j C w_{t-j} \end{aligned}$$

and

$$y_t = G x_t = G A_0^t x_0 + G \sum_{j=0}^{t-1} A_0^j C w_{t-j}.$$

**Impulse response functions**: Either representation has $A_0^j C$ as the response of $x_{t+1}$ to $w_{t+1-j}$ at each lag $j$. E.g. a contribution of a shock $w_{t-j}$ to $x_t$ is $A_0^j C$. Also, its contribution to $y_t$ is $G A_0^j C$.

## 3. Forecasting

Markov property. Time-$t$ conditional forecast:

$$E_t x_{t+j} = A_0^j x_t$$

where $E_t x_{t+1} = E\left(x_{t+1} | x_t, x_{t-1}, ..., x_0\right)$.

If $y_t = G x_t$ and we have

$$E_t \sum_{j=0}^{\infty} \beta^j y_{t+j} = G \left(I - \beta A_0\right)^{-1} x_t.$$

provided $\beta A_0$ has eigenvalues less than 1 in modulus.

**EXERCISE 1**. *Suppose our state space model is AR*$(1)$*,*

$$x_{t+1} = \rho x_t + \varepsilon_{t+1}$$
$$y_t = \beta x_t, \quad |\rho| < 1$$

*where* $x_t, y_t, \varepsilon_t \in \mathbb{R}$.

*Compute*

1. *the stationary mean, variance, and the autocovariance function of* $\{y_t\}$ .
2. *the impulse response function for* $\{y_t\}$.

## REFERENCES

LJUNGQVIST, L., AND T. J. SARGENT (2004): *Recursive Macroeconomic Theory*. MIT Press, Cambridge, MA.