ON STATIONARY RECURSIVE EQUILIBRIA AND NON-DEGENERATE STATE SPACES: THE HUGGETT MODEL

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ABSTRACT

The seminal work of Huggett ["The risk-free rate in heterogeneous-agent incomplete-insurance economies", *Journal of Economic Dynamics and Control*, 1993, 17(5-6), 953-969] showed that in a stationary recursive equilibrium, there exists a unique stationary distribution of agent types. However, the question remains open if an equilibrium's individual state space might turn out to be such that either: (i) every agent's common borrowing constraint binds forever, so that the distribution of agents will be degenerate; or (ii) that the individual state space might be unbounded. By invoking a simple comparative-static argument, we provide closure to this open question. We show that the equilibrium individual state space must be compact and that this set has positive measure. From Huggett's result that there is a unique distribution of agents in a stationary equilibrium, our result implies that it must also be one that is nontrivial or nondegenerate.

KEYWORDS: Incomplete markets; Compactness; Individual state space; Stationary distribution
JEL CODES: C62; D31; D52

1. Introduction

The seminal quantitative heterogeneous-agent framework of Huggett [1993] describes a stationary recursive competitive equilibrium by a Markov operator on an endogenous compact space of probability measures. Huggett [1993, Theorem 2] proved that for a given (stationary equilibrium) interest rate, there is a unique distribution of agents as a fixed point of this Markov operator.[1]

To establish the second result [Huggett, 1993, Theorem 2] certain sufficiency conditions in Theorem 2 of Hopenhayn and Prescott [1992] are required to be satisfied by the model. One of the sufficient conditions is compactness of agents' equilibrium individual state space, denoted by S. Huggett [1993] showed that in a stationary recursive equilibrium, each agent indexed by an asset-endowment pair $(a, e) \in S$ would always remain in the set S.

In this note, we amend the proof of Lemma 3 in Huggett [1993] and show that in a stationary recursive equilibrium, the individual state space must be compact and of a positive measure. We do so by invoking a simple comparative statics argument. That is, we provide the rationale for an implicit assumption used in proving Lemma 3 of Huggett [1993]. This implies that a stationary recursive equilibrium distribution cannot be degenerate. Our result affirms what Huggett [1993] showed in terms of the existence of a stationary recursive equilibrium and the corresponding stability property of its equilibrium distribution.

More generally, one may ask if a sequential competitive equilibrium exists; and if so whether there is a (payoff equivalent) recursive equilibrium associated with a given sequential equilibrium.[2] Krebs [2004] considered an incomplete-markets exchange economy, but with a finite number of consumers with "nonbinding endogenous borrowing (short-sale) constraints".[3] Krebs [2004, Proposition 2] showed that there is no recursive equilibrium given by time-invariant continuous functions over a compact extended equilibrium state space, if these constraints never bind. For a recursive equilibrium to exist, markets must be effectively complete or borrowing constraints must be effectively ad-hoc [Krebs, 2004, Proposition 1]. The latter case relates to our focus on the Huggett [1993] economy with exogenous borrowing constraints. This is also the case assumed in Miao [2006].[4]

Also, Miao [2006, Assumption 5] introduced a uniform upper bound on labor allocations to guarantee an equilibrium upper bound on assets (via market clearing conditions and regularity assumptions on the production function). However, we extend the proof of Huggett [1993, Lemma 3] that there is a finite endogenous upper bound on agents' asset state space, to guarantee that it must be larger than the borrowing-constraint lower bound. In contrast to Krebs [2004] and Miao [2006], we directly restrict attention to a stationary recursive equilibrium, since the main result

[1] In the setting of Huggett [1993], the key insight on the risk-free rate anomaly arising from the perspective of representative agent models, was obtained by way of incomplete asset markets (single non-state contingent asset) and precautionary saving motives arising from an exogenous borrowing constraint. This framework is one of the key foundations for further quantitative research using heterogenous agent macroeconomics. In this class of models, important questions such as asset pricing puzzles [see e.g. Huggett, 1993; Aiyagari, 1994], and fiscal policy and taxation [see e.g. Heathcote, 2005], can be addressed.

[2] A sequential (competitive) equilibrium is characterised in terms of infinite-sequences of allocations and relative prices satisfying the requirements of general equilibrium.

[3] These are called endogenous constraints in that they only rule out Ponzi schemes and nothing else.

[4] Miao [2006] proved the existence of a sequential competitive equilibrium in an economy with a continuum of heterogeneous agents and finite-state-space aggregate shocks. Miao [2006] showed that for any sequential competitive equilibrium, there exists a payoff-equivalent recursive equilibrium representation.

of Miao [2006, Theorem 3] on the existence of a recursive equilibrium from a given sequential equilibrium can be adapted to the special-case Huggett [1993] model.

A related literature, although in the context of the Brock and Mirman [1972] stochastic growth model, is also concerned with strengthening results on existence and uniqueness of nontrivial stationary distributions [see e.g. Chatterjee and Shukayev, 2008; Kamihigashi, 2006] while relaxing the number of hypotheses required. The enterprise in this paper is related to this literature, but in the context of the Huggett model.

2. HUGGETT'S MODEL

In this section we first revisit the Huggett [1993] model. Then we provide a brief discussion on the notion of an endogenously compact individual state space and its implication for the existence of a unique distribution of agents in a stationary recursive equilibrium. Hereinafter, we will use the term "stationary equilibrium" synonymously with "stationary recursive equilibrium".

In the Huggett model, time is discrete, and each period is indexed by $t \in \mathbb{N} := \{0, 1, ...\}$.[5] The population of agents has mass 1. Each measure zero agent receives a stream of stochastic endowment of consumption good. Let $E = \{e_l, e_h\}$, where $e_h > e_l$, be the set of endowment realizations. Each random sequence $(e_t)_{t \in \mathbb{N}}$ is governed by a given Markov chain (π, π_0) on E, where π is the stochastic matrix and $\pi_0 \in \Delta(E)$ is the initial unconditional distribution on E that belongs in the simplex $\Delta(E)$. $\pi(e'|e) := \Pr\{e_{t+1} = e'|e_t = e\} > 0$, $e', e \in E$, is independent of t, and any other agent's realization of e. Let $A := [\underline{a}, +\infty)$ be the space of possible asset levels. The parameter $\underline{a}$ is interpreted as an exogenous borrowing constraint. Denote the product individual state space as $X := A \times E$.

2.1. An individual's decision problem.

The *individual state* is $x := (a, e) \in X$. The individual takes as given the aggregate price $q > 0$. Suppose in an equilibrium there is a set $S := [\underline{a}, \bar{a}] \times \{e_l, e_h\} \subset X$. Denote the Borel σ-algebra generated by open subsets of S as $\mathcal{B}(S)$. The dependence of the equilibrium q on the *aggregate state* given by a probability measure ψ on $(S, \mathcal{B}(S))$ is implicit.[6]

Each agent has a common subjective discount factor $\beta \in (0, 1)$, and identical per-period utility $u : \mathbb{R}_+ \to \mathbb{R}$. The function u is strictly increasing, strictly concave, and twice continuously differentiable. Each agent chooses consumption (c) and saving in terms of a single asset (a'). Let the agent's feasible action correspondence be $\Gamma(q) : A \times E \to \mathcal{B}(\mathbb{R}_+ \times A)$, where at each slice of Γ indexed by $(x; q)$, we have a description of the feasible choice set of an agent *currently* named x:

$$\Gamma(x; q) = \left\{(c, a') : a + e \geq c + a'q, \;\; c \geq 0, \;\; a' \geq \underline{a}\right\}.$$

Denote $(x; q) \mapsto v(x; q) \in \mathbb{R}$ as an agent's value function. Each agent's Bellman equation is

$$v(x; q) = \max_{(c,a') \in \Gamma(x;q)} \left\{ u(c) + \beta \sum_{e' \in E} v(a', e'; q)\pi(e'|e) \right\}, \tag{1}$$

with associated optimal decision rule $(x; q) \mapsto \hat{a}(x; q)$, such that at $(x; q)$, $a' = \hat{a}(x; q)$.

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[5] As is the usual convention, we may drop the explicit time-t subscript on variables, e.g. $x := x_t$ and $x' := x_{t+1}$.

[6] In Huggett [1993], since the emphasis is on a notion of *recursive stationary equilibrium* where q is constant, we don't have to explicitly carry around the distribution of agent types ψ, as a relevant state variable. Instead, we only make the agents' problems dependent on q as a scalar parameter.

2.2. **Compact equilibrium individual state space $S \subset X$.** The notion of a stationary equilibrium is defined in Huggett [1993, p.956]. Given the individual endowment processes' Markov operator, $\pi : \Delta(E) \to \Delta(E)$, and, an optimal decision rule, $(x; q) \mapsto \hat{a}(x; q)$, we can induce a time-invariant probability measure ψ on the measurable space $(S, \mathcal{B}(S))$ satisfying an equilibrium Markov operator:

$$\psi(B) = \int_S P(x, B) \mathrm{d}\psi, \qquad \forall B \in \mathcal{B}(S), \tag{2}$$

where $P : S \times \mathcal{B}(S) \to [0, 1]$ is the equilibrium transition probability function.[7]

In Theorem 1, Huggett [1993] provides some sufficient conditions on the model such that given q, the solution to each agent's Bellman equation problem has some nice properties. Specifically, Theorem 1 in Huggett [1993] establishes that the optimal $\hat{a} : X \to [\underline{a}, \infty)$ is *continuous*, is either *strictly increasing* in a, if $a > \underline{a}$, or is *nondecreasing* in a if $a = \underline{a}$.

Theorem 1 and Lemmata 1-3 in Huggett [1993] show that each agent's optimal decision function for credit holdings, $\hat{a} : X \to [\underline{a}, \infty)$, has the typical shape as in Figure 1. In particular, this decision rule has the following properties:

(1) If the current endowment is e_l, and, if the borrowing constraint is not binding, then $\hat{a}(\cdot, e_l)$ is well below the 45°-line in (a, a')-space. That is, an agent who is not currently credit constrained in terms of his asset choice for the following period, and who continues to face a sequence of low endowment realizations, will be reducing his asset level in each corresponding subsequent periods (Lemma 1) until his asset level hits the borrowing constraint, $\underline{a}$, where either the agent is constrained to just borrow $\underline{a}$ in the subsequent period; or
(2) If the agent has high endowment, e_h, the agent will start saving, but there is an asset level, $\overline{a}$, such that the policy function at e_h, $\hat{a}(\cdot, e_h)$, crosses the 45°-line in (a, a')-space. This is proved by Lemma 3, which uses both Lemma 1 and Lemma 2.

Thus, in an equilibrium, if there is to be an endogenous $\overline{a}$, as shown in Lemma 3 in Huggett [1993], which is the smallest fixed point satisfying $\hat{a}(a, e_h) = a$, then it is straightforward to deduce that $S := [\underline{a}, \overline{a}] \times \{e_l, e_h\}$ is an endogenously compact metric space. That is, each agent x beginning in S will always stay within S, or the equilibrium asset decision rule will be $\hat{a} : S \to [\underline{a}, \overline{a}]$.

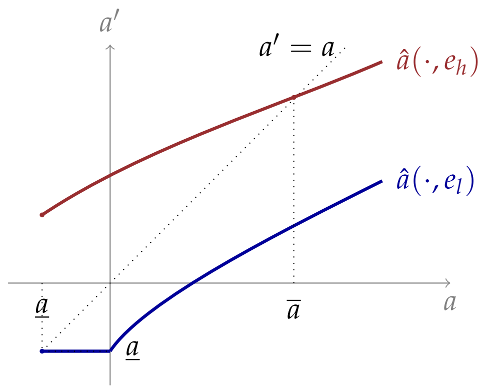


FIGURE 1. Characterization of optimal policy function $\hat{a} : X \to \mathbb{R}$. When $e = e_h$, for each a, $\hat{a}(a, e_h) > \hat{a}(a, e_l)$. If $\hat{a}(a, e) \geq \underline{a}$ binds, then $\hat{a}(\cdot, e)$ is nondecreasing in a.

[7]The details are discussed very nicely in Huggett [1993].

Theorem 2 of Huggett, applying Theorem 2 in Hopenhayn and Prescott [1992], provides sufficient conditions and shows (by convergence in the weak topology) the existence of a unique distribution ψ satisfying (2), for a given q consistent with a stationary equilibrium. These conditions in turn include the requirement that S is a compact metric space.

3. A Further Exposition

Huggett's Lemma 3 states that under the conditions of his Theorem 2, there exists a fixed point $\overline{a}$ satisfying $\hat{a}(a, e_h) = a$.[8] To prove this result, Huggett supposed the contrapositive that there does not exist any a such that $\hat{a}(a, e_h) = a$. Note that this implies also that at $\underline{a}$, we have $\hat{a}(\underline{a}, e_h) > \underline{a}$. From this, Huggett obtained the proof by contradiction.

In this section, we expound on Huggett's results and show that the endogenous (*i.e.* stationary equilibrium) upper bound $\overline{a}$ on assets is nontrivial. That is, we show that $\underline{a} < \overline{a} < \infty$. In doing so, we also provide a rationale for why Huggett made only one contrary hypothesis in his proof of Lemma 3 in Huggett [1993], and in particular, why $\hat{a}(\underline{a}, e_h) > \underline{a}$. This implication from Lemma 3 in Huggett is crucial toward a result that in a stationary equilibrium of his model, there exists a *unique* distribution of agent types.

If we consider a stationary equilibrium (as in Huggett) with a unique distribution of agents, then implicit in this is the requirement that $\underline{a}$ itself *cannot* be a fixed point satisfying $\hat{a}(\underline{a}, e_h) = \underline{a}$. That is, if we are interested in a stationary equilibrium $S := [\underline{a}, \overline{a}] \times E$ where $\underline{a} \neq \overline{a}$, then a contrary hypothesis to this must consider three possible cases:

H1. $\hat{a}(a, e_h) < a$ for $a > \underline{a}$ and $\hat{a}(a, e_h) = a$ for $a = \underline{a}$,
H2. $\hat{a}(a, e_h) > a$ for $a > \underline{a}$ and $\hat{a}(a, e_h) = a$ for $a = \underline{a}$, and
H3. There is no a such that $\hat{a}(a, e_h) = a$.

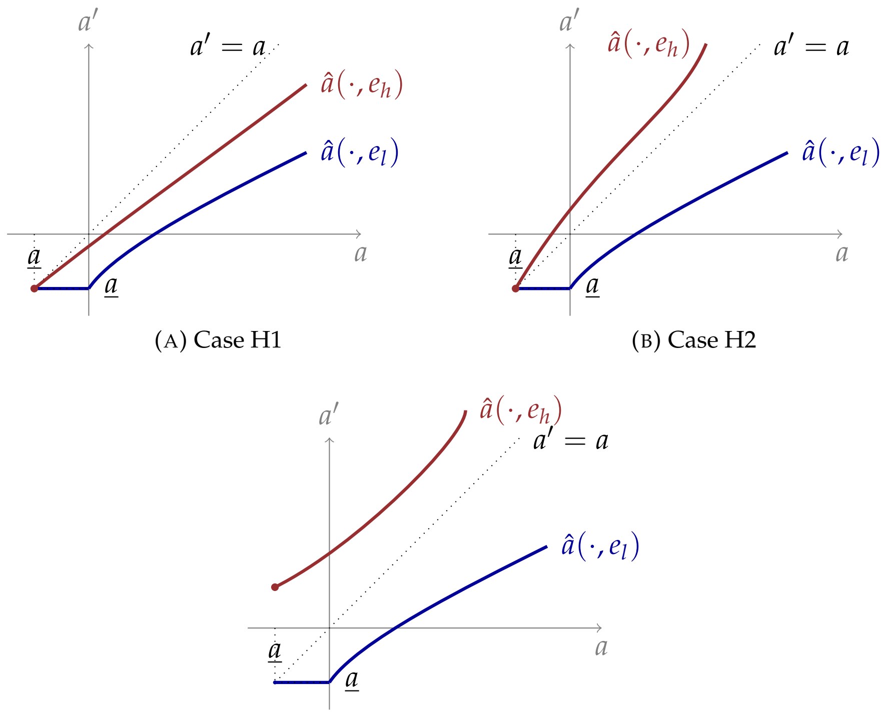
These three contrary hypotheses are depicted by the typical graphs in Figure 2. In establishing the result on the existence of an endogenously compact S by contradiction in Lemma 3, Huggett [1993] made only the contrary hypothesis (H3) that there is no a such that $\hat{a}(a, e_h) = a$.

It turns out, from our Lemma 1 below, that only one of these contrary hypotheses is possible (i.e. H3), as was assumed in Huggett [1993]. We thus provide a further exposition and confirmation of Huggett's proof to his Lemma 3. Moreover, in conjunction with Huggett's Lemma 3, we can further deduce (from Theorem 1 below) that in a stationary equilibrium, the endogenous $S := [\underline{a}, \overline{a}] \times \{e_l, e_h\}$ must be a nontrivial compact metric space, so that the equilibrium probability measure of agents on the measure space $(S, \mathcal{B}(S))$ is unique and nondegenerate.

Lemma 1. *Assume the conditions in Theorem 2 of Huggett [1993]. In a* ***stationary equilibrium*** *with equilibrium price of credit q, each agent's optimal decision rule $\hat{a}(\cdot, \cdot; q) : S \to [\underline{a}, \infty)$ must be such that $\hat{a}(\underline{a}, e_h; q) > \hat{a}(\underline{a}, e_l; q)$.*

Proof. Suppose to the contrary that in a stationary equilibrium at price q, we have $\hat{a}(\underline{a}, e_h) \leq \hat{a}(\underline{a}, e_l)$. Since by construction, it is not possible to have $\hat{a}(\underline{a}, e) < \underline{a}$ for all $e \in E$, then it suffices to suppose to the contrary that $\hat{a}(\underline{a}, e_h) = \hat{a}(\underline{a}, e_l)$ in a stationary equilibrium. We want to show that $\hat{a}(\underline{a}, e_h) = \hat{a}(\underline{a}, e_l)$ contradicts some requirements of stationary equilibrium. To do so, we will characterize

[8] These conditions are: (i) $0 < \beta < q$; (ii) $a + e_l - \underline{a}q > 0$; and (iii) $\pi(e_h|e_h) \geq \pi(e_h|e_l)$. Hereinafter, we suppress the explicit dependency of each agent's decision on the constant price q. Hence, $\hat{a}(a, e)$ should be understood to mean $\hat{a}(a, e; q)$, where q is the price of credit in a stationary equilibrium.



(A) Case H1

(B) Case H2

(C) Case H3

FIGURE 2. In proving Lemma 3 in Huggett [1993], suppose there is no a $(> \underline{a})$ such that $\hat{a}(a, e_h) = a$. *A priori* there may be three possible cases for the component function $\hat{a}(\cdot, e_h)$ that would satisfy this hypothesis. We can rule out cases H1 and H2.

individuals' credit demand behavior under this hypothesis; first, at the current states where agents begin with maximal borrowing, and second, at the states where they do not.

First, consider agents at the current state $(\underline{a}, e)$ for each $e \in E$. Denote the consumption at the current maximal credit $\underline{a}$ in each endowment state e_l and e_h as $c_l^{\underline{a}}$ and $c_h^{\underline{a}}$, respectively. These are, respectively, given by the per-period budget constraints as

$$c_l^{\underline{a}} = \underline{a} + e_l - \underline{a}q,$$

and,

$$c_h^{\underline{a}} = \underline{a} + e_h - \underline{a}q.$$

The last terms on the RHS of each constraint above are, respectively, due to the fact that $\hat{a}(\underline{a}, e_l) = \underline{a}$, as implied by Theorem 1 and Lemma 1 in Huggett [1993], and $\hat{a}(\underline{a}, e_h) = \hat{a}(\underline{a}, e_l)$ by hypothesis. We now characterize the stationary equilibrium behavior of agents at $(\underline{a}, e)$, where $e \in E$, by induction. Consider the current period $t \geq 0$ for an agent whose state is $(\underline{a}, e_l)$. Since $\hat{a}(\underline{a}, e_l) = \underline{a}$, equilibrium current consumption at this realized state $(\underline{a}, e_l)$ is $c_l^{\underline{a}}$. The Kuhn-Tucker conditions for this agent yield

$$u_c(c_l^{\underline{a}}) \geq \beta q^{-1} \sum_{e' \in E} \pi(e'|e_l) u_c[\underline{a} + e' - \hat{a}(\underline{a}, e')q]. \tag{3}$$

If the agent is currently $(\underline{a}, e_h)$, then, current consumption is $c_h^{\underline{a}}$. Since by hypothesis, this agent optimally demands credit at $\hat{a}(\underline{a}, e_h) = \underline{a}$, the Kuhn-Tucker conditions for this agent give

$$u_c(c_h^{\underline{a}}) \geq \beta q^{-1} \sum_{e' \in E} \pi(e'|e_h) u_c[\underline{a} + e' - \hat{a}(\underline{a}, e')q]. \tag{4}$$

Beginning from $t+1$, it must be that these agents at $(\underline{a}, e')$ will remain choosing $\hat{a}(\underline{a}, e') = \underline{a}$, for any realization of $e' \in E$, since $\hat{a}(\underline{a}, e_l) = \underline{a}$, and the hypothesis that $\hat{a}(\underline{a}, e_h) = \hat{a}(\underline{a}, e_l)$, are optimal and stationary equilibrium selections at the respective states. This implies that equilibrium consumption in period $t+1$ will again, either be $c_l^{\underline{a}}$ if the individual's state is $(\underline{a}, e_l)$, or, $c_h^{\underline{a}}$ if the individual becomes $(\underline{a}, e_l)$. Then the respective Kuhn-Tucker weak inequalities in (3) and (4) carry over to $t+1$. The same optimal choices will be made beginning in any subsequent period $\tau \geq t+1$. In other words, these agents beginning from $(\underline{a}, e)$ are stuck forever rolling over their debt at the maximal credit level $\underline{a}$, regardless of the realization of their stochastic endowment each period, $e \in E$.

Second, consider any other agent beginning at some (a, e), where $a > \underline{a}$. By Theorem 1 in Huggett [1993], $\hat{a}(a, e)$ is strictly increasing in a for all (a, e) at a given (here stationary equilibrium) q, such that $\hat{a}(a, e; q) > \underline{a}$. Consider the function $\hat{a}(\cdot, e_l)$. By Lemma 1 in Huggett [1993], $\hat{a}(a, e_l) < a$ for $a > \underline{a}$. Pick any agent beginning at (a, e_l), where $a > \underline{a}$. Define a sequence $\{w_n\}$ by $w_1 = a$, $w_2 = \hat{a}(w_1, e_l)$, $w_3 = \hat{a}(w_2, e_l)$, ... Then, $\{w_n\} \to \underline{a}$ monotonically. Now, consider $\hat{a}(a, e_h)$. There are four possible cases.

- C.1 $\hat{a}(a, e_h) > a$ for all $a > \underline{a}$. Define a sequence $\{x_n\}$ by $x_1 = a$, $x_2 = \hat{a}(x_1, e_h)$, $x_3 = \hat{a}(x_2, e_h)$, ... Then, $\{x_n\} \to \infty$ monotonically.
- C.2 $\hat{a}(a, e_h) < a$ for some $a \in (\underline{a}, a^*)$, where $a^* > \underline{a}$, and, $\hat{a}(a, e_h) > a$ for some $a \in (a^*, \infty)$. If $a \in (\underline{a}, a^*)$, then the agent beginning from (a, e_h), and receiving endowment e_h forever, will follow a similar monotone sequence as $\{y_n\} \to \underline{a}$. If $a \in (a^*, \infty)$, then the agent beginning from (a, e_h), and receiving endowment e_h forever, will follow a similar monotone sequence as $\{x_n\} \to \infty$. The agent beginning at (a^*, e_h), and receiving endowment e_h forever, remains there forever. However, this agent at (a^*, e_h), like all other agents elsewhere, will receive e_l with positive probability $\pi(e_l|e_h)$ next period. Hence (a^*, e_h) is not a stable attractor.
- C.3 $\hat{a}(a, e_h) > a$ for some $a \in (\underline{a}, a^*)$, where $a^* > \underline{a}$, and, $\hat{a}(a, e_h) < a$ for some $a \in (a^*, \infty)$. This case has two fixed points: one at $\underline{a}$ and the other at a^* such that $\hat{a}(a^*, e_h) = a^*$.
- C.4 $\hat{a}(a, e_h) < a$ for all $a > \underline{a}$. Define a sequence $\{y_n\}$ by $y_1 = a$, $y_2 = \hat{a}(y_1, e_h)$, $y_3 = \hat{a}(y_2, e_h)$, ... Then, $\{y_n\} \to \underline{a}$ monotonically.

Now we can consider the implications of all the possible cases C.1–C.4. From the (supposedly) equilibrium behavioral characterization above, and, if cases C.1 and C.2 above hold, then we can deduce that the equilibrium individual state space $S = [\underline{a}, \infty) \times E$ is not compact, since $[\underline{a}, \infty)$ is not closed and bounded. This violates a sufficient condition for the existence of a unique stationary probability measure of agents: Huggett [1993, Theorem 2] and Hopenhayn and Prescott [1992, Theorem 2]. If Case C.3 is true in a stationary equilibrium, this implies that there are two stationary probability measures on the compact set $S = [\underline{a}, a^*] \times E$. One of the two is degenerate and cannot be consistent with equilibrium – i.e. if the economy began with all agents holding asset $\underline{a}$ then all agents remain there forever, since Case C.3 assumed that $\hat{a}(\underline{a}, e_h) = \underline{a}$. However, in this case the excess demand for credit is exactly $\underline{a} < 0$; it violates credit market clearing which requires zero excess demand for credit, i.e., $\int_S \hat{a}(a,e;q)d\psi = 0$. Therefore $\hat{a}(\underline{a}, e_h) > \underline{a}$ in this case.

If case C.4 prevails, then all agents will be optimally demanding the maximal credit level $\underline{a}$. However, in this case, it also violates credit market clearing which requires zero excess demand for credit, i.e., $\int_S \hat{a}(a,e;q)d\psi = 0$. since the excess demand for credit in Case C.4 is exactly $\underline{a} < 0$.

Therefore, the hypothesis that $\hat{a}(\underline{a}, e_h) = \hat{a}(\underline{a}, e_l)$ is a stationary equilibrium optimal selection contradicts the definition of a stationary equilibrium. In a stationary equilibrium, agents' best responses at any $(\underline{a}, e)$ must then have the property that $\hat{a}(\underline{a}, e_h) > \hat{a}(\underline{a}, e_l)$. □

Lemma 1 above thus provides the reason why it suffices to make only one case (*i.e.* Case H3 in Figure 2) of a contrary hypothesis when proving Huggett's Lemma 3. This leads us to the following theorem that reinforces the arguments summarized in Lemma 3 of Huggett [1993].

Theorem 1. *Under the assumptions of Theorem 2 in Huggett [1993], the decision rule $\hat{a}(a, e_h; q)$ in a stationary equilibrium has a unique fixed point $a^* < \infty$, i.e., $a(a^*, e_h; q) = a^*$, and $a^* \neq \underline{a}$.*

Proof. By Theorem 1 in Huggett, $\hat{a}(\cdot,\cdot;q) : [\underline{a}, \infty) \times E \to [\underline{a}, \infty)$ is continuous on $[\underline{a}, \infty)$ for each $e \in E$, strictly increasing in a for all (a, e) such that $\hat{a}(a, e; q) > \underline{a}$. Combining our Lemma above, which says $\hat{a}(\underline{a}, e_h; q) > \underline{a}$, and Lemma 3 in Huggett [1993], which states the existence of a fixed point a^* satisfying $\hat{a}(a, e_h; q) = a$, we can then conclude that $\underline{a} < a^* < \infty$. Then, let $\bar{a} = a^*$. □

Therefore, in a stationary equilibrium, the endogenous $S := [\underline{a}, \bar{a}] \times \{e_l, e_h\}$ must be a nontrivial compact metric space. Our Theorem then implies that in a stationary equilibrium, there is a unique distribution of agents in the Huggett model which is nondegenerate. This is useful *a priori* information when numerically computing approximate solutions to this class of models.

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